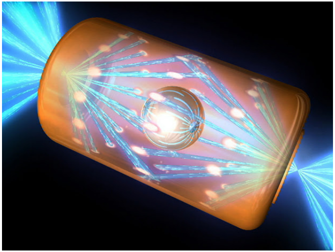
Modélisation et méthodes numériques pour l'étude du transport de particules dans un plasma

Sébastien Guisset

Soutenance de thèse de l'Université de Bordeaux
Talence, le 23 Septembre 2016

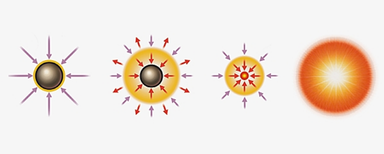


Directeur de thèse : **Stéphane Brull**, Institut Mathématiques de Bordeaux
Co-Directeur de thèse : **Emmanuel d'Humières**, Laboratoire Celia

Physical context

↪ Contribution to the modelling and numerical methods for the transport of charged particles in plasmas

↪ Hot plasmas created by lasers



General context: Understanding of the processes leading to ignition of the fusion reactions by inertial confinement

Multiphysics processes:

- Laser-plasma absorption
- Neutron production
- Radiative transfer
- Transport of particles

Related research areas:

- Hypersonic flows
- Radiotherapy
- Magnetic confinement fusion
- Astrophysics

↪ long time regimes studies (hydrodynamics scales)

Outline

1. Modelling in plasma physics: the angular moments models

2. Numerical methods for the study of the particle transport on large scales

3. First step towards multi-species modelling: the angular M_1 model in a moving frame

4. Conclusion / Perspectives

Modelling

Plasma: set of totally ionised atoms. Electronic transport, fixed ions

Kinetic description: electron distribution function $f(t, x, v)$,

$\hookrightarrow$ Resolution of the Vlasov or Fokker-Planck-Landau equation

$$\underbrace{\frac{\partial f}{\partial t} + v.\nabla_x f}_{advection\ \ term} + \underbrace{\frac{q_\alpha}{m_\alpha}(E + v \times B).\nabla_v f}_{force\ \ \ term} = \underbrace{C_{ee}(f, f) + C_{ei}(f),}_{collisional\ \ \ terms}$$

Accurate but numerically expensive (usually limited to short scales)

Hydrodynamic description: cheap but less accurate for far equilibrium regimes

$\hookrightarrow$ describe kinetic effects on fluid time scales is challenging!

$\hookrightarrow$ Intermediate description, angular moment models.

Angular moments models

$\hookrightarrow$ Angular moments extraction: $v = \zeta\, \Omega$ with $\zeta = |v|$.

$$f_0(\zeta) = \zeta^2 \int_{S_2} f(v) d\Omega, \quad f_1(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega d\Omega, \quad f_2(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega \otimes \Omega d\Omega.$$

Set of admissible states[1]

$$\mathcal{A} = \Big((f_0, f_1) \in \mathbb{R} \times \mathbb{R}^3, \;\; f_0 \geq 0, \quad |f_1| \leq f_0 \Big).$$

Angular moments model

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \dfrac{q}{m}\partial_\zeta (f_1.E) = 0, \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \dfrac{q}{m}\partial_\zeta (f_2 E) - \dfrac{q}{m\zeta}(f_0 E - f_2 E) - \dfrac{q}{m}(f_1 \wedge B) = 0. \end{cases}$$

$\hookrightarrow$ Closure relation?

[1] D. Kershaw, Tech. Report (1976).

The P_N closure

Spherical Harmonic expansion[2]

$$f(t,x,\zeta,\Omega) = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} A_n^m f_n^m(t,x,\zeta) Y_n^m(\Omega),$$

with

$$Y_n^m(\Omega) = P_n^{|m|}(\cos\theta) e^{im\varphi}, \qquad A_n^m = \frac{(2n+1)(n-|m|)!}{(n+|m|)!},$$

and $P_n^m(z)$ are the associated Legendre functions[3].

$\hookrightarrow$ Positivity of the distribution function is required

$\hookrightarrow$ Positive P_N closure [4]

$\hookrightarrow$ We prefer a closure based on a entropy minimisation criterion[5]

---

2 Pomraning, Pergamon Press (1973).
3 Abramowitz and Stegun. Dover Publications (1964).
4 Hauck and McLarreen. Siam J. Sci. Comput. (2010).
5 G.N. Minerbo, J. Quant. Spectrosc. Ra. (1978).

The M_1 closure

Determination of f_2 as a function of f_0 and f_1: Entropy minimisation problem[6,7].

$$\min_{f\geq 0}\left\{ \ \mathcal{H}(f) \ / \ \forall \zeta \in \mathbb{R}^+, \ \ \zeta^2 \int_{S^2} f(\Omega,\zeta) d\Omega = f_0(\zeta), \ \ \zeta^2 \int_{S^2} f(\Omega,\zeta)\Omega d\Omega = f_1(\zeta) \ \right\},$$

$$\text{with } \ \mathcal{H}(f) = \zeta^2 \int_{S^2} (f \ \ln \ f \ - \ f) d\Omega.$$

Entropy minimisation principle[8]:

$$f(\Omega,\zeta) = \exp(a_0(\zeta) + a_1(\zeta).\Omega) \geq 0,$$

- positivity
- hyperbolicity
- entropy dissipation

Expression of f_2:

$$f_2 = \left(\frac{1-\chi(\alpha)}{2} Id + \frac{3\chi(\alpha)-1}{2} \frac{f_1}{|f_1|} \otimes \frac{f_1}{|f_1|} \right) f_0,$$

with

$$\chi(\alpha) = \frac{1+|\alpha|^2+|\alpha|^4}{3}, \qquad \alpha = f_1/f_0.$$

[6] G.N. Minerbo, J. Quant. Spectrosc. Ra. (1978).
[7] D. Levermore, J. Stat. Phys. (1996).
[8] B. Dubroca and J.L. Feugeas. C. R. Acad. Sci. Paris Ser. I (1999).

Advantages and limitations of the M_1 model

Advantages

- Intermediate models (compromise)
- Application to radiative transfer and radiotherapy
- Accurate for isotropic configurations or configurations with one dominant direction[9]

Limitations

- Validity of angular moments models for kinetic plasma studies?
- Complex configurations in collisionless regimes[10] (not presented here, see chapter 2)

$\hookrightarrow$ Adapted for collisional plasma applications

[9] Dubroca, Feugeas and Frank. Eur. Phys. J. (2010).

[10] Guisset, Moreau, Nuter, Brull, d'Humières, Dubroca, Tikhonchuk. J. Phys. A Math. Theor. (2015).

Collisional operators

Electronic Fokker-Planck-Landau equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f + \frac{q}{m}(E + v \times B).\nabla_v f = C_{ee}(f,f) + C_{ei}(f),$$

$$C_{ee}(f,f) = \alpha_{ee} div_v \Big(\int_{v' \in \mathbb{R}^3} S(v - v')[\nabla_v f(v) f(v') - f(v) \nabla_v f(v')] dv' \Big),$$

$$C_{ei}(f) = \alpha_{ei} div_v \Big(S(v) \nabla_v f(v) \Big), \quad S(u) = \frac{1}{|u|^3}(|u|^2 Id - u \otimes u).$$

$\hookrightarrow$ C_{ee} non-linear: complex angular moments extraction

Simplification

$$C_{ee}(f,f) \approx Q_{ee}(F_0) = C_{ee}(F_0, F_0)^{11,12} \qquad F_0 = \frac{f_0}{\zeta^2} = \int_{S^2} f d\Omega.$$

$\hookrightarrow$ Angular moments extraction

[11] Berezin, Khudick and Pekker J. Comput. Phys. (1987).
[12] Buet and Cordier J. Comput. Phys. (1998).

Collisional operators

Electronic M_1 model:

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta \left(\frac{qE}{m} f_1 \right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta \left(\frac{qE}{m} f_2 \right) - \frac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1). \end{cases}$$

Collision operators

$$Q_0(f_0) = \alpha_{ee}\partial_\zeta \Big(\zeta^2 A(\zeta) \partial_\zeta (\frac{f_0}{\zeta^2}) - \zeta B(\zeta) f_0 \Big), \qquad Q_1(f_1) = -\alpha_{ei}\frac{2f_1}{\zeta^3},$$

$$A(\zeta) = \int_0^\infty \min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^2 f_0(\mu) d\mu, \quad B(\zeta) = \int_0^\infty \min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^3 \partial_\mu (\frac{f_0(\mu)}{\mu^2}) d\mu.$$

$\hookrightarrow$ Admissibility requirement

Modification: admissible M_1 model[13]

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta \left(\frac{qE}{m} f_1 \right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta \left(\frac{qE}{m} f_2 \right) - \frac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1). \end{cases}$$

[13] J. Mallet, S. Brull and B. Dubroca. KRM (2015).

Collisional operators

Fundamental properties of the M_1 collisional operators[14,15]:

- admissibility
- H-theorem (entropy dissipation)
- conservation properties
- caracterisation of the equilibrium states

$\hookrightarrow$ Long time behavior: derivation of the plasma transport coefficients

Boltzmann $\rightarrow$ Chapman-Enskog expansion: Navier-Stokes

Fokker-Planck-Landau $\rightarrow$ Spitzer-Härm approximation: Electron collisional hydrodynamics

Electronic M_1 model $\rightarrow$ Spitzer-Härm approximation: Electron collisional hydrodynamics

$\hookrightarrow$ different plasma transport coefficients

[14] Mallet, Brull, Dubroca. KRM (2015)

[15] Guisset, Brull, Dubroca, d'Humières, Tikhonchuk. Physica A (2016).

Electron collisional hydrodynamics

Strongly collisional fully ionised hot plasma:

$$f(t,x,\zeta,\Omega) = \mathcal{M}_f(\zeta,\, T_e(t,x), n_e(t,x)) + \varepsilon F(t,x,\zeta,\Omega),$$

where $\varepsilon = \lambda_{ei}/L$,

$$\mathcal{M}_f(\zeta,\, T_e(t,x), n_e(t,x)) = n_e(t,x)\Big(\frac{m_e}{2\pi T_e(t,x)}\Big)^{3/2} \exp\Big(-\frac{m_e\zeta^2}{2T_e(t,x)}\Big),$$

$$F(t,x,\zeta,\Omega) = F_0(t,x,\zeta) + F_1(t,x,\zeta).\Omega.$$

Density and energy conservation laws:

$$\left\{\begin{array}{l} \dfrac{\partial n_e}{\partial t} + \nabla_x.(n_e u_e) = 0, \\ \dfrac{\partial T_e}{\partial t} + u_e.\nabla_x(T_e) + \dfrac{2}{3}T_e\nabla_x.(u_e) + \dfrac{2}{3n_e}\nabla_x.(q) = \dfrac{2}{3n_e}j.E, \end{array}\right.$$

where

$$j = -en_e u_e = -\frac{4\pi e}{3}\int_0^{+\infty} F_1\zeta^3 d\zeta, \qquad q = \frac{2\pi}{3}\int_0^{+\infty} F_1(m_e\zeta^2 - 5T_e)\zeta^3 d\zeta.$$

$\hookrightarrow$ Closure: derivation of F_1

Plasma transport coefficients

Long time behavior

$$\mathcal{M}_f \zeta \Big(\frac{eE^*}{T_e} + \frac{1}{2T_e} \nabla_x (T_e) \big(\frac{m_e \zeta^2}{T_e} - 5 \big) \Big) = -\frac{2\alpha_{ei}}{\zeta^3} F_1 + \frac{1}{\zeta^2} Q_0(\zeta^2 F_1),$$

with

$$E^* = E + (1/en_e)\nabla_x (n_e T_e).$$

↪ Solve an integro-differential equation[16]

↪ Expansion[17,18] of F_1 on the generalised Laguerre polynomials

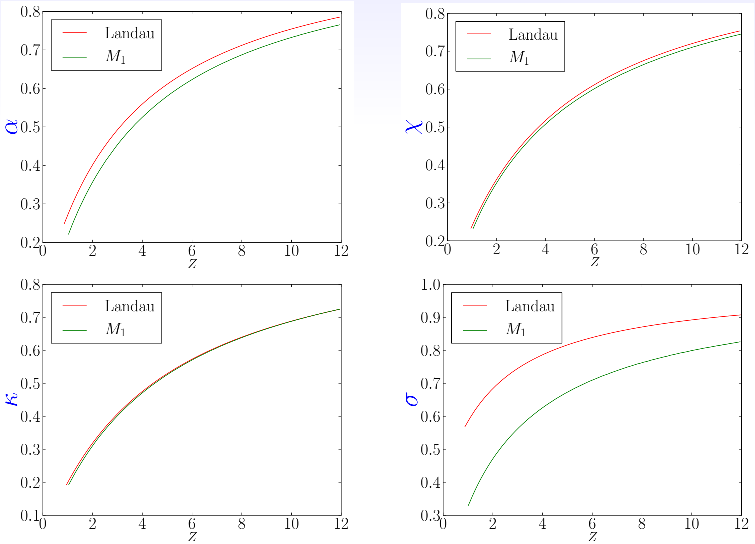
Closure

$$j = \sigma E^* + \alpha \nabla_x T_e, \quad q = -\alpha T_e E^* - \chi \nabla_x T_e.$$

---

[16] L. Spitzer and R. Härm. Phys. Rev. (1953).
[17] S.I. Braginskii. Rev. Plasma Phys. (1965).
[18] S. Chapman. Phil. Trans. Roy. Soc. (1916).

Plasma transport coefficients[19]



$\hookrightarrow$ Very good agreement for α, χ, κ and good agreement for σ.

[19] Guisset, Brull, Dubroca, d'Humières, Tikhonchuk. Physica A (2016).

Outline

1. Modelling in plasma physics: the angular moments models

2. Numerical methods for the study of the particle transport on large scales

3. First step towards multi-species modelling: the angular M_1 model in a moving frame

4. Conclusion / Perspectives

Long time behavior and singular limit

- Different scales:

$$\lambda_{De}, \tau_{pe} \quad << \quad \lambda_{ei}, \tau_{ei} \quad << \quad L, T$$

Quasi-neutral limit ($t^* >> \tau_{pe}$)

$$\begin{cases} \dfrac{\partial f}{\partial t} + v.\nabla_x f - (E + v \times B).\nabla_v f = C_{ee}(f,f) + C_{ei}(f), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\alpha^2}, \end{cases}$$

with $\alpha = \tau_{pe}/t^*$.

Diffusive limit[20] ($t^* >> \tau_{ei}$)

$$\begin{cases} \varepsilon \dfrac{\partial f}{\partial t} + v.\nabla_x f - (E + v \times B).\nabla_v f = \dfrac{1}{\varepsilon} C_{ee}(f,f) + \dfrac{1}{\varepsilon} C_{ei}(f), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\varepsilon^3 \alpha^2}, \end{cases}$$

with $\varepsilon^2 = \tau_{ei}/t^*$. $\hookrightarrow$ constraints on the numerical schemes

$\hookrightarrow$ asymptotic-preserving schemes

[20] S. Bianchini, B. Hanouzet and R. Natalini. Commun. Pur. Appl. Math. (2007).

Quasi-neutral limit ($t^* >> \tau_{pe}$)

As $\alpha \to 0$, impossibility to compute E^{n+1}.

$\hookrightarrow$ Reformulation[21,22] of the M_1-Maxwell model[23]

Time semi-discretisation

$$\frac{f_1^{n+1} - f_1^n}{\Delta t} + \nabla_x(\zeta f_2^n) - \partial_\zeta(E^{n+1} f_2^n) + \frac{E^{n+1}}{\zeta}(f_0^n - f_2^n) = Q_0(f_1^n) + Q_1(f_1^n).$$

Electric current: $$j^n = -\int_0^{+\infty} f_1^n \zeta d\zeta,$$

$$\begin{cases} \dfrac{j^{n+1} - j^n}{\Delta t} = \beta_1(f_0^n, f_1^n) E^{n+1} + \beta_2(f_0^n, f_1^n), \\ \dfrac{E^{n+1} - E^n}{\Delta t} = -\dfrac{j^{n+1}}{\alpha^2}. \end{cases} \quad E^{n+1} = \frac{-\dfrac{\alpha^2 E^n}{\Delta t^2} + \beta_2(f_0^n, f_1^n) + \dfrac{j^n}{\Delta t}}{-\dfrac{\alpha^2}{\Delta t^2} - \beta_1(f_0^n, f_1^n)}.$$

If $\alpha \to 0$ we can obtain E^{n+1}, Δt is not constrained by α (asymptotic stability).

$\hookrightarrow$ Realistic collision operators

[21] Degond, Liu, Savelief, Vignal J. Sci. Comp. (2012).
[22] Degond, Deluzet, Savelief. J. Comp. Phys. (2012).
[23] Guisset, Brull, d'Humières, Dubroca, Karpov, Potapenko. CICP (2016).

Diffusive limit[24] $(t^* >> \tau_{ei})$

Diffusive scaling: $\tilde{t} = t/t^*, \quad \tilde{x} = x/x^*, \quad \tilde{v} = v/v_{th},$
such that $\tau_{ei}/t^* = \varepsilon^2$, $\lambda_{ei}/x^* = \varepsilon$.

Dimensionless system (Lorentz approximation)

$$\begin{cases} \varepsilon \partial_t f_0 + \zeta \partial_x f_1 + E \partial_\zeta f_1 = 0, \\ \varepsilon \partial_t f_1 + \zeta \partial_x f_2 + E \partial_\zeta f_2 - \dfrac{E}{\zeta}(f_0 - f_2) = -\dfrac{2\sigma}{\zeta^3} \dfrac{f_1}{\varepsilon}. \end{cases}$$

Hilbert expansion

$$f_0^\varepsilon = f_0^0 + \varepsilon f_0^1 + O(\varepsilon^2), \quad f_1^\varepsilon = f_1^0 + \varepsilon f_1^1 + O(\varepsilon^2).$$

Limit equation $$f_1^0 = 0,$$

$$\partial_t f_0^0 + \zeta \partial_x \left(-\frac{\zeta^4}{6\sigma} \partial_x f_0^0 - \frac{E\zeta^3}{6\sigma} \partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0 \right) + E \partial_\zeta \left(-\frac{\zeta^4}{6\sigma} \partial_x f_0^0 - \frac{E\zeta^3}{6\sigma} \partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0 \right) = 0.$$

$\hookrightarrow$ Mixed x and ζ derivatives

[24] Collaboration with R. Turpault

Derivation of the scheme: problem setting

- Simplified case: no electric field

Model and diffusive limit

$$\begin{cases} \varepsilon\partial_t f_0^\varepsilon + \zeta\partial_x f_1^\varepsilon = 0, \\ \varepsilon\partial_t f_1^\varepsilon + \zeta\partial_x f_2^\varepsilon = -\dfrac{2\sigma}{\zeta^3}\dfrac{f_1^\varepsilon}{\varepsilon}. \end{cases} \qquad \partial_t f_0^0(t,x) - \zeta\partial_x\left(\frac{\zeta^4}{6\sigma(x)}\partial_x f_0^0(t,x)\right) = 0.$$

Limit of the HLL scheme:

$$\begin{cases} \varepsilon\dfrac{f_{0i}^{n+1,\varepsilon} - f_{0i}^{n,\varepsilon}}{\Delta t} + \zeta\dfrac{f_{1i+1}^{n,\varepsilon} - f_{1i-1}^{n,\varepsilon}}{2\Delta x} - a_x\dfrac{f_{0i+1}^{n,\varepsilon} - 2f_{0i}^{n,\varepsilon} + f_{0i-1}^{n,\varepsilon}}{\Delta x} = 0, \\ \varepsilon\dfrac{f_{1i}^{n+1,\varepsilon} - f_{1i}^{n,\varepsilon}}{\Delta t} + \zeta\dfrac{f_{2i+1}^{n,\varepsilon} - f_{2i-1}^{n,\varepsilon}}{2\Delta x} - a_x\dfrac{f_{1i+1}^{n,\varepsilon} - 2f_{1i}^{n,\varepsilon} + f_{1i-1}^{n,\varepsilon}}{\Delta x} = -\dfrac{2\sigma_i}{\zeta^3}\dfrac{f_{1i}^{n+1,\varepsilon}}{\varepsilon}. \end{cases}$$

$\hookrightarrow$ HLL numerical scheme: numerical viscosity in $O(\dfrac{\Delta x}{\varepsilon})$, wrong limit.

$\hookrightarrow$ New scheme to compute f_{0i}^{n+1}

Harten, Lax and van Leer formalism

Riemann problem for hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0,$$

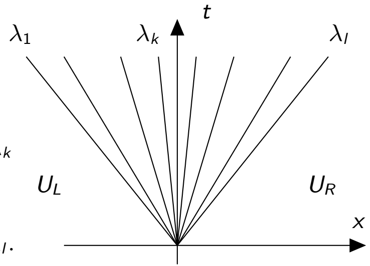
with $U \in \mathbb{R}^m, x \in \mathbb{R}, t > 0$. Initial conditions

$$U(x, t=0) = \begin{cases} U_L & \text{if} \quad x < 0, \\ U_R & \text{if} \quad x > 0. \end{cases}$$

$\hookrightarrow$ Self-similarity of the exact Riemann solution $U(x/t, U_L, U_R)$

Approximate Riemann solver

$$U_{RP}(x/t, U_L, U_R) = \begin{cases} U_1 = U_L & \text{if} \quad x/t < \lambda_1, \\ \vdots \\ U_k & \text{if} \quad \lambda_{k-1} < x/t < \lambda_k \\ \vdots \\ U_{l+1} = U_R & \text{if} \quad x/t > \lambda_l. \end{cases}$$



Harten, Lax and van Leer formalism

Godunov-type scheme

$$U_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U^h(x, t^{n+1})dx,$$

$$U^h(x, t^{n+1}) = U_{RP}\Big(\frac{x - x_{i+1/2}}{t^n + \Delta t}, U_i, U_{i+1}\Big) \ \text{ if } \ x \in [x_i, x_{i+1}].$$

Consistency with the integral form of the hyperbolic system[25]

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_0^{\Delta t}(\partial_t U + \partial_x F(U))dxdt = 0,$$

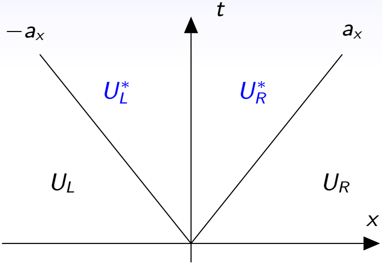
$$\hookrightarrow F(U_R) - F(U_L) = \sum_{k=1}^{l} \lambda_k(U_{k+1} - U_k).$$

[25] A. Harten, P. Lax, B. Van Leer. Siam Review, 1983.

Approximate Riemann solver

Source terms: approximate Riemann solvers which own a stationnary discontinuity[26,27] (0-contact discontinuity).

$$U_{\mathcal{R}}(x/t) = \begin{cases} U^L & \text{if } x/t < -a_x \\ U^{L*} & \text{if } -a_x < x/t < 0 \\ U^{R*} & \text{if } 0 < x/t < a_x \\ U^R & \text{if } a_x < x/t \end{cases}$$



$\hookrightarrow$ Two intermediate states $U^{L*} = {}^t(f_0^{L*}, f_1^*)$ and $U^{R*} = {}^t(f_0^{R*}, f_1^*)$.

CFL condition $$\Delta t \leq \frac{\Delta x}{2a_x}.$$

Numerical scheme

$$f_{0i}^{n+1} = \frac{a_x \Delta t}{\Delta x} f_{0i-1/2}^{R*} + (1 - \frac{2a_x \Delta t}{\Delta x}) f_{0i}^n + \frac{a_x \Delta t}{\Delta x} f_{0i+1/2}^{L*}.$$

---

[26] F. Bouchut, Frontiers in Mathematics series (2004).
[27] L. Gosse, Math. Mod. Meth. Apl. Sci. (2001)

Derivation of f_1^*

Consistency condition for f_1^*

$$f_1^* = \frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L) - \frac{2}{\zeta^3}\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x)f_1(x,t)dtdx.$$

Approximation (stiff source term)

$$\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x)f_1(x,t)dtdx \approx \bar{\sigma}\Delta t f_1^*, \qquad \bar{\sigma} = \sigma(0).$$

Definition of f_1^*, [28]

$$f_1^* = \frac{2a_x\zeta^3}{2a_x\zeta^3 + 2\bar{\sigma}\Delta x}\left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L)\right].$$

[28] C. Berthon and R.Turpault. Numer. Meth. Part. D. E. (2011).

Derivation of f_0^{L*} and f_0^{R*}

Consistency condition for f_0

$$\frac{f_0^{L*} + f_0^{R*}}{2} = \frac{f_0^L + f_0^R}{2} - \frac{1}{2a_x}(\zeta f_1^R - \zeta f_1^L).$$

Definition of f_0^{L*} and f_0^{R*}

$$\begin{cases} f_0^{L*} = \tilde{f}_0 - \Gamma, \\ f_0^{R*} = \tilde{f}_0 + \Gamma. \end{cases} \qquad \tilde{f}_0 = \frac{f_0^L + f_0^R}{2} - \frac{1}{2a_x}(\zeta f_1^R - \zeta f_1^L).$$

The Rankine-Hugoniot conditions gives Γ

$$\begin{cases} f_0^{L*} = f_0^L - \frac{\zeta}{a_x}(f_1^* - f_1^L), \\ f_0^{R*} = f_0^R - \frac{\zeta}{a_x}(f_1^R - f_1^*). \end{cases} \qquad \Gamma = \frac{1}{2}[f_0^R - f_0^L - \frac{\zeta}{a_x}(f_1^L - 2f_1^* + f_1^R)].$$

Admissibility conditions: modification of f_0^{L*} and f_0^{R*}

$$\begin{cases} f_0^{L*} = \tilde{f}_0 - \Gamma\theta, \\ f_0^{R*} = \tilde{f}_0 + \Gamma\theta. \end{cases} \qquad \tilde{\theta} = \frac{\tilde{f}_0 - |f_1^*|}{|\Gamma|} \geq 0, \qquad \theta = \min(\tilde{\theta}, 1).$$

Properties of the scheme

- Admissibility requirement

If $U_{i,j}^n \in \mathcal{A}$ then $U_{i,j}^{n+1} \in \mathcal{A}$ under the CFL condition $\Delta t \leq \Delta x/(2a_x)$.

- Consistency in the limit regime

In the diffusive regime, the unknown $f_{0i}^{n+1,0}$ satisfies the discrete equation

$$\frac{f_{0i}^{n+1,0} - f_{0i}^{n,0}}{\Delta t} - \frac{\zeta}{\Delta x}\left(\frac{\zeta^4}{6\bar{\sigma}_{i+1/2}\Delta x}(f_{0i+1}^{n,0} - f_{0i}^{n,0}) - \frac{\zeta^4}{6\bar{\sigma}_{i-1/2}\Delta x}(f_{0i}^{n,0} - f_{0i-1}^{n,0})\right) = 0.$$

$\hookrightarrow$ In the limit $\theta = 1$, no limitation is required

$\hookrightarrow$ Homogeneous case with electric field
source term naturally included and well-balanced property

General model

Model

$$\begin{cases} \varepsilon\partial_t f_0 + \zeta\partial_x f_1 + E\partial_\zeta f_1 = 0, \\ \varepsilon\partial_t f_1 + \zeta\partial_x f_2 + E\partial_\zeta f_2 - \dfrac{E}{\zeta}(f_0 - f_2) = -\dfrac{2\sigma}{\zeta^3}\dfrac{f_1}{\varepsilon}. \end{cases}$$

Diffusive limit

$$\partial_t f_0^0 + \zeta\partial_x\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma}f_0^0\right) + E\partial_\zeta\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma}f_0^0\right) = 0.$$

Consider

$$\begin{aligned} \frac{f_{0ij}^{n+1} - f_{0ij}^n}{\Delta t} =& \frac{a_x}{\Delta x} f_{0i-1/2j}^{R*} - \frac{2a_x}{\Delta x} f_{0ij}^n + \frac{2a_x\Delta t}{\Delta x} f_{0i+1/2j}^{L*} \\ &+ \frac{a_\zeta}{\Delta\zeta} f_{0ij-1/2}^{R*} - \frac{2a_\zeta}{\Delta\zeta} f_{0ij}^n + \frac{a_\zeta}{\Delta\zeta} f_{0ij+1/2}^{L*}. \end{aligned}$$

$\hookrightarrow$ Wrong asymptotic limit, mixed derivatives?

General model

Mixed derivatives: Modification of the intermediate state f_1^*

$$f^*_{1i+1/2j} = \alpha_{i+1/2j}\Big(\frac{f_{1i+1j}+f_{1ij}}{2} - \frac{1}{2a_x}(\zeta_j f_{2i+1j} - \zeta_j f_{2ij}) - c_{i+1/2j}\theta_{1i+1/2j}(\frac{\partial f_0}{\partial \zeta})_{i+1/2j}(1-\alpha_{i+1/2j})\Big)$$

$$f^*_{1ij+1/2} = \beta_{ij+1/2}\Big(\frac{f_{1ij+1}+f_{1ij}}{2} - \frac{1}{2a_\zeta}(E_i f_{2ij+1} - E_i f_{2ij}) - \bar{c}_{ij+1/2}\theta_{2ij+1/2}(\frac{\partial f_0}{\partial x})_{ij+1/2}(1-\beta_{ij+1/2})\Big)$$

with

$$\alpha_{i+1/2j} = \frac{2a_x\zeta_j^3}{2a_x\zeta_j^3 + \sigma_{i+1/2}\Delta x}, \qquad \beta_{ij+1/2} = \frac{2a_\zeta\zeta_{j+1/2}^3}{2a_\zeta\zeta_{j+1/2}^3 + \sigma_i\Delta\zeta}.$$

$\hookrightarrow$ c and $\bar{c}$ are fixed to obtain the correct limit equation

$$c_{i+1/2j} = \frac{E_{i+1/2}\Delta x}{3a_x}, \qquad \bar{c}_{ij+1/2} = \frac{\zeta_{j+1/2}\Delta\zeta}{3a_\zeta}.$$

General model

Upwinding depending of the sign of $c_{i+1/2j}$ and $\bar{c}_{ij+1/2}$

$$\bar{c}_{ij+1/2}(\frac{\partial f_0}{\partial x})_{ij+1/2} \approx \bar{c}_{ij+1/2}\frac{f_{0i+1j+1} - f_{0ij+1} + f_{0i+1j} - f_{0ij}}{2\Delta x},$$

$$c_{i+1/2j}(\frac{\partial f_0}{\partial \zeta})_{i+1/2j} \approx \begin{cases} c_{i+1/2j}\frac{f_{0i+1j} - f_{0i+1j-1} + f_{0ij} - f_{0ij-1}}{2\Delta\zeta} & \text{if } c_{i+1/2j} < 0, \\ c_{i+1/2j}\frac{f_{0i+1j+1} - f_{0i+1j} + f_{0ij+1} - f_{0ij}}{2\Delta\zeta} & \text{if } c_{i+1/2j} > 0. \end{cases}$$

$\hookrightarrow$ $\theta_{1i+1/2j}$ and $\theta_{2ij+1/2}$ fixed to ensure the admissibility conditions.

- Admissibility property

If $U_{i,j}^n \in \mathcal{A}$ then $U_{i,j}^{n+1} \in \mathcal{A}$ under the CFL condition $\Delta t \leq \dfrac{\Delta\zeta\Delta x}{2a_x\Delta\zeta + 2a_\zeta\Delta x}$.

- Consistency in the limit regime

Numerical test cases: diffusive regime

Initial conditions

$$f_0(0,x,\zeta) = \begin{cases} 1 & \text{if } x \leq L/3, \\ 0 & \text{if } L/3 \leq x \leq 2L/3, \\ 1 & \text{if } L/3 \leq x, \end{cases} \qquad f_1(0,x,\zeta) = 0.$$

Periodical boundary conditions, $\alpha_{ei} = 10^4$ and $E = 0$.

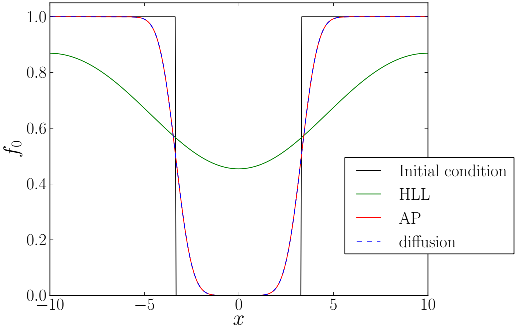


Figure: diffusive regime: f_0 profile at time t=200.

Numerical test cases: diffusive regime

Periodical boundary conditions, $\alpha_{ei} = 10^4$ and $E = 1$.

Initial conditions

$$\begin{cases} f_0(0, x, \zeta) = \zeta^2 \exp(-x^2) \exp(-2(\zeta - 3)^2), \\ f_1(0, x, \zeta) = 0. \end{cases}$$

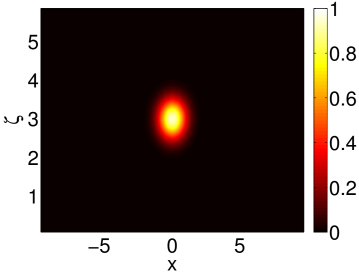


Figure: f_0 profile at initial time.

Numerical test cases: diffusive regime

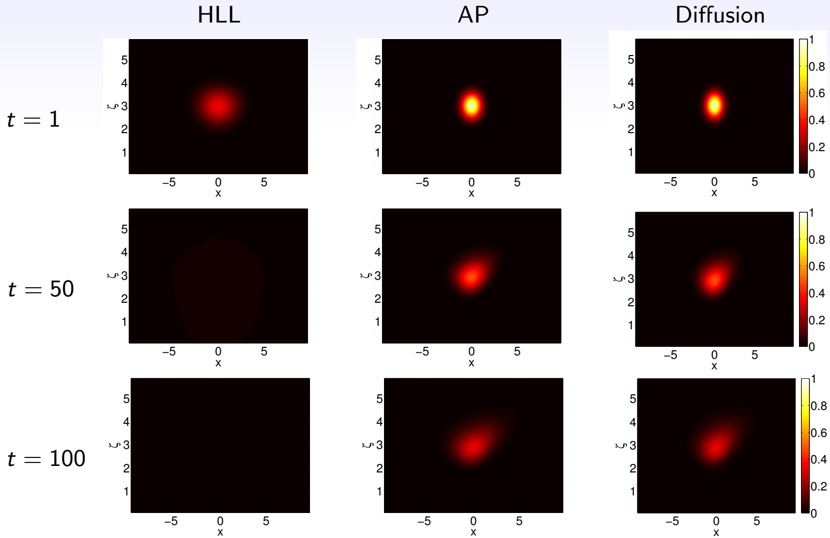
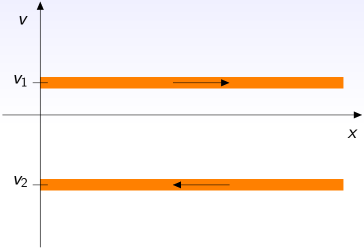


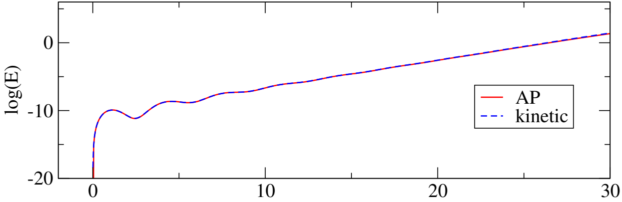
Figure: f_0 profile at time t=1 (top), t=50 (middle), t=100 (bottom), for the HLL scheme (left), the AP scheme (middle) and the diffusion equation (right), ($\alpha_{ei} = 10^4$).

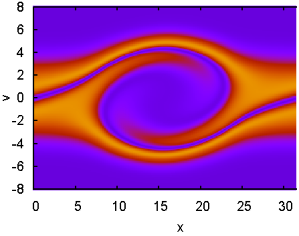
Numerical test cases: Electron beams interaction



Initial conditions:

$$f(0,x,v) = \frac{1}{2}(1 + A\cos(kx))\exp(-(v+v_1)^2) + \frac{1}{2}(1 - A\cos(kx))\exp(-(v+v_2)^2),$$
$$E(0,x) = 0.$$





Duclous, Dubroca, Filbet and Tikhonchuk. J. Comput. Phys. (2009)

Numerical test cases: non-constant collisional parameter

Linear profile: $\alpha_{ei}(x) = ax + b, \quad \alpha_{ei}(x_{min} = -40) = 5 \cdot 10^3, \quad \alpha_{ei}(x_{max} = 40) = 10^5.$

Self-consistent electric field: $E = -\partial_x T.$

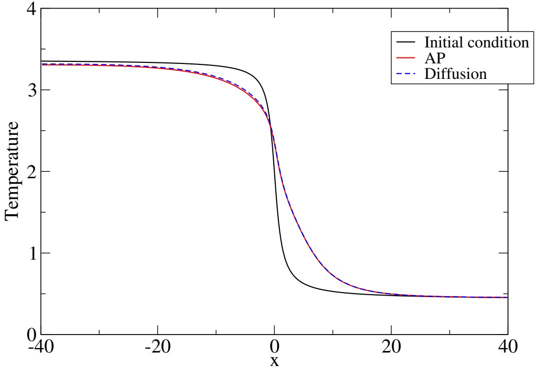


Figure: Temperature profile at time t=5000.

Outline[29]

[29] Collaboration with D. Aregba-Driollet

First step towards multispecies modelling

$\hookrightarrow$ Change of velocity frame[30,31]

Interests:

- Velocity grid reduction
- Simplification of the collisional operators
- Galilean invariance property for angular moments models

Velocity change of variable

$$c = v - u(t,x), \qquad f(t,x,v) = g(t,x,c).$$

Kinetic equations $\quad \partial_t f + div_x(vf) = C(f),$

$$\hookrightarrow \ \partial_t g + div_x((c+u)g) - div_c((\partial_t u + \partial_x u(c+u))g) = C(g).$$

$\hookrightarrow$ Evolution equation for u ?

30 F. Filbet and T. Rey J. Comp. Phys. (2013)
31 A. Bobylev, J. Carrillo and I. Gamba. J. Stat. Phys. (2000).

First step towards multispecies modelling

Different choices of u:

- Ion mean velocity

$$\partial_t(n^i u^i)+div_x(\int_{\mathbb{R}^3} f^i(v)v\otimes v\,dv)-q^i n^i E = \int_{\mathbb{R}^3} C_{ie}(v)vdv, \quad \frac{1}{n}\int_{\mathbb{R}^3} g(c)cdc = u^e - u^i.$$

$\hookrightarrow$ Plasma physics applications

First step: one species of non-charged particles

- Particles mean velocity (main difficulties of the multispecies case)

Momentum equation

$$\partial_t(nu) + div_x(nu\otimes u + \int_{\mathbb{R}^3} g(c)c\otimes c\,dc) = 0, \qquad \int_{\mathbb{R}^3} g(c)cdc = 0.$$

$\hookrightarrow$ Rarefied gas dynamics applications

$\hookrightarrow$ Heat flux

Angular moments model

$\hookrightarrow$ Angular moments extraction

The M_1 angular moment model

$$\begin{cases} \partial_t g_0 + div_x(\zeta g_1 + u g_0) - \partial_\zeta\left(\dfrac{du}{dt}.g_1 + \zeta \partial_x u : g_2\right) = 0, \\ \partial_t g_1 + div_x(\zeta g_2 + u g_1) - \partial_\zeta\left(g_2 \dfrac{du}{dt} + \zeta g_3 \partial_x u\right) + \dfrac{g_0 Id - g_2}{\zeta}\dfrac{du}{dt} + \left(\partial_x u g_1 - g_3 \partial_x u\right) = 0, \end{cases}$$

where $\dfrac{du}{dt} = \partial_t u + (\partial_x u)u.$

Evolution equation on u

$$\partial_t u + (\partial_x u)u + \frac{1}{n} div_x\left(\int_0^{+\infty} g_2(\zeta)\zeta^2 d\zeta\right) = 0.$$

$\hookrightarrow$ Closure relation for g_2 and g_3.

Model properties

The M_1 angular moment model

$$\begin{cases} \partial_t g_0 + div_x(\zeta g_1 + u g_0) - \partial_\zeta\left(\frac{du}{dt}.g_1 + \zeta\partial_x u : g_2\right) = 0, \\ \partial_t g_1 + div_x(\zeta g_2 + u g_1) - \partial_\zeta\left(g_2\frac{du}{dt} + \zeta g_3 \partial_x u\right) + \frac{g_0 Id - g_2}{\zeta}\frac{du}{dt} + \left(\partial_x u g_1 - g_3\partial_x u\right) = 0, \end{cases}$$

with

$$\partial_t u + (u.\partial_x)u + \frac{1}{n} div_x\left(\int_0^{+\infty} g_2(\zeta)\zeta^2 d\zeta\right) = 0.$$

- Godunov's symmetrisation[32] (entropic variables[33]): Friedrichs-symmetric system
- Conservation laws
  - Mass and energy conservation
  - Momentum conservation and zero mean velocity

$$\int_0^{+\infty} g_1(\zeta)\zeta d\zeta = 0.$$

- Galilean invariance (rotational and translational invariance)

---

[32] T. Goudon, C. Lin. J. Math. Anal. Appl. (2013).
[33] D. Levermore. J. Stat. Phys. (1996).

Galilean invariance

A_0 : fixed frame

$$\partial_t f + div_x(vf) = C(f)$$

Galilean invariance

$$\tilde{x} = x - st$$
$$\tilde{v} = v - s$$

A_0 : fixed frame $\longrightarrow$ B_0 : Uniform translation frame

$$\partial_t f + div_x(vf) = C(f)$$

$$\partial_t \tilde{f} + div_{\tilde{x}}(\tilde{v}\tilde{f}) = C(\tilde{f})$$

Galilean invariance

$$\tilde{x} = x - st$$

$$\tilde{v} = v - s$$

A_0 : fixed frame $\longrightarrow$ B_0 : Uniform translation frame

$$\partial_t f + div_x(vf) = C(f)$$

$$\partial_t \tilde{f} + div_{\tilde{x}}(\tilde{v}\tilde{f}) = C(\tilde{f})$$

$$c = v - u(t,x)$$

$$nu(t,x) = \int_v fvdv$$

$\downarrow$

A : Mobile frame

$$\partial_t g + div_x((c+u)g) - div_c\Big[(\partial_t u + \partial_x u(c+u))g\Big] = C(g)$$

Galilean invariance

$$\tilde{x} = x - st$$
$$\tilde{v} = v - s$$

A_0 : fixed frame $\longrightarrow$ B_0 : Uniform translation frame

$$\partial_t f + div_x(vf) = C(f)$$

$$\partial_t \tilde{f} + div_{\tilde{x}}(\tilde{v}\tilde{f}) = C(\tilde{f})$$

$$c = v - u(t,x)$$
$$nu(t,x) = \int_v fvdv$$

$$\tilde{c} = \tilde{v} - \tilde{u}(t,\tilde{x})$$
$$n\tilde{u}(t,\tilde{x}) = \int_{\tilde{v}} \tilde{f}\tilde{v}d\tilde{v}$$

A : Mobile frame

B : Uniform translation
Mobile frame

$$\partial_t g + div_x((c+u)g) - div_c\Big[(\partial_t u + \partial_x u(c+u))g\Big] = C(g)$$

$$\partial_t \tilde{g} + div_{\tilde{x}}((\tilde{c}+\tilde{u})\tilde{g}) - div_{\tilde{c}}\Big[(\partial_t \tilde{u} + \partial_{\tilde{x}} \tilde{u}(c+\tilde{u}))\tilde{g}\Big] = C(\tilde{g})$$

Galilean invariance

$$\tilde{x} = x - st$$
$$\tilde{v} = v - s$$

A_0 : fixed frame $\longrightarrow$ B_0 : Uniform translation frame

$$\partial_t f + div_x(vf) = C(f)$$

$$\partial_t \tilde{f} + div_{\tilde{x}}(\tilde{v}\tilde{f}) = C(\tilde{f})$$

$$c = v - u(t,x)$$
$$nu(t,x) = \int_v fvdv$$

$$\tilde{c} = \tilde{v} - \tilde{u}(t,\tilde{x})$$
$$n\tilde{u}(t,\tilde{x}) = \int_{\tilde{v}} \tilde{f}\tilde{v}d\tilde{v}$$

$$\tilde{x} = x - st$$
$$\tilde{c} = c$$
$$\tilde{u} = u - s$$

A : Mobile frame $\longrightarrow$ B : Uniform translation Mobile frame

$$\partial_t g + div_x((c+u)g) - div_c\left[(\partial_t u + \partial_x u(c+u))g\right] = C(g)$$

$$\partial_t \tilde{g} + div_{\tilde{x}}((\tilde{c}+\tilde{u})\tilde{g}) - div_{\tilde{c}}\left[(\partial_t \tilde{u} + \partial_{\tilde{x}} \tilde{u}(c+\tilde{u}))\tilde{g}\right] = C(\tilde{g})$$

Numerical schemes

One-dimension spatial framework with a BGK collision operator

$$\begin{cases} \partial_t g_0 + \partial_x(\zeta g_1 + u g_0) - \partial_\zeta\left(\dfrac{du}{dt} g_1 + \zeta \partial_x u g_2\right) = \dfrac{1}{\tau}(M_{g_0} - g_0), \\ \partial_t g_1 + \partial_x(\zeta g_2 + u g_1) - \partial_\zeta\left(\dfrac{du}{dt} g_2 + \zeta \partial_x u g_3\right) + \dfrac{du}{dt}\dfrac{g_0 - g_2}{\zeta} + \partial_x u\Big(g_1 - g_3\Big) = -\dfrac{1}{\tau} g_1, \end{cases}$$

with

$$M_{g_0} = \zeta^2 4\pi n \Big(\frac{m}{2\pi T}\Big)^{\frac{3}{2}} \exp(-\frac{m\zeta^2}{2T}),$$

and

$$n = \int_0^{+\infty} g_0 d\zeta, \qquad T = \frac{m}{3n}\int_0^{+\infty} g_0 \zeta^2 d\zeta.$$

$\hookrightarrow$ Suitable numerical scheme?

Numerical schemes

Intermediate state

$$\begin{cases} \partial_t g_0 - \partial_\zeta(\frac{du}{dt} g_1 + \zeta \partial_x u g_2) = 0, \\ \partial_t g_1 - \partial_\zeta(\frac{du}{dt} g_2 + \zeta \partial_x u g_3) = 0. \end{cases}$$

Underlying kinetic equation

$$\partial_t g(\zeta) + \partial_\zeta(b(\zeta) g(\zeta)) = 0,$$

with $b(\zeta) = -(\frac{du}{dt}\mu + \zeta \partial_x u \mu^2).$

$\hookrightarrow$ Variable coefficient scalar equation

Conservative scheme

$$\frac{g_j^{n+1} - g_j^n}{\Delta t} + \frac{h_{j+1/2}^n - h_{j-1/2}^n}{\Delta \zeta} = 0,$$

with $h_{j+1/2}^n = b_{j+1/2}^- g_{j+1}^n + b_{j+1/2}^+ g_j^n,$ $\qquad b^\pm = \frac{1}{2}(b \pm |b|).$

Numerical scheme

Angular moments extraction

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{G_{j+1/2}^n - G_{j-1/2}^n}{\Delta \zeta} = 0,$$

with

$$U_j^{n+1} = \begin{pmatrix} g_{0j}^{n+1} \\ g_{1j}^{n+1} \end{pmatrix}, \qquad G_{j+1/2}^n = \frac{1}{2}\Big[\frac{du}{dt}\begin{pmatrix} g_{1j+1}^n + g_{1j}^n \\ g_{2j+1}^n + g_{2j}^n \end{pmatrix} + \zeta_{j+1/2}\partial_x u \begin{pmatrix} g_{0j+1}^n + g_{0j}^n \\ g_{1j+1}^n + g_{1j}^n \end{pmatrix}$$
$$+(|\frac{du}{dt}| + ||\zeta||_\infty |\partial_x u|)\begin{pmatrix} g_{0j+1}^n - g_{0j}^n \\ g_{1j+1}^n - g_{1j}^n \end{pmatrix}\Big].$$

$\hookrightarrow$ Admissibility requirement under CFL condition

- Same procedure for the spatial derivative
- Standard discretisation for the source terms and collisional terms

$\hookrightarrow$ Admissibility for the complete scheme under reduced CFL condition

Numerical result: temperature gradient

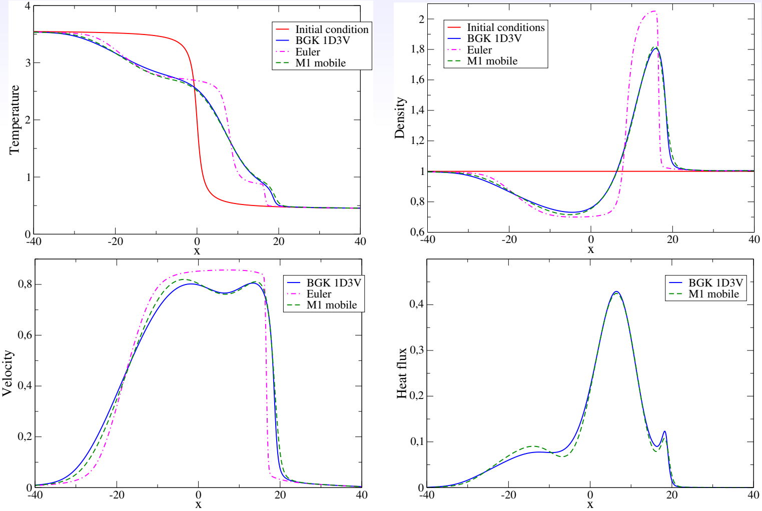


Figure: Temperature gradient with variable Knudsen at time $t = 10$.
$\tau(x) = \frac{1}{2}(\arctan(1 + 0.1x) + \arctan(1 - 0.1x))$.

Conclusion

Plasma physics modelling

- Better understanding of the validity regimes of M_1 and M_2 angular moments models for collisionless plasma applications
- Appropriate collisional operators for the electronic M_1 model

Adapted numerical methods for the study of the large scale particles transport

- Study of numerical resolution of the electronic M_1 model in the quasi-neutral limit
- Study of numerical resolution of the electronic M_1 model in the diffusive limit

First step towards multi-species modelling

- Angular M_1 model in a moving frame

Perspectives

Modelling in plasma physics

- Transport coefficients considering magnetic fields[34]

Numerical methods for the study of the large scale particles transport

- Extension for electron-electron collision operator
- Uniform consistency / high order extensions
- Modified HLL schemes for diffusive regimes[35]
- Asymptotic-preserving scheme for the quasi-neutral and diffusive limits

Angular M_1 model in a moving frame

- Discrete energy conservation / discrete zero mean velocity
- Motion of heavy and light particles (ions and electrons)
- Easier and more systematic derivation of Galilean invariant minimum entropy moment systems [36]

---

[34] Braginskii, Reviews of Plasma Physics (1965).
[35] Collaboration with C. Chalons.
[36] M. Junk and A. Unterreiter, Cont. Mech. Thermodyn. (2002).

Thank you

Uniform consistency

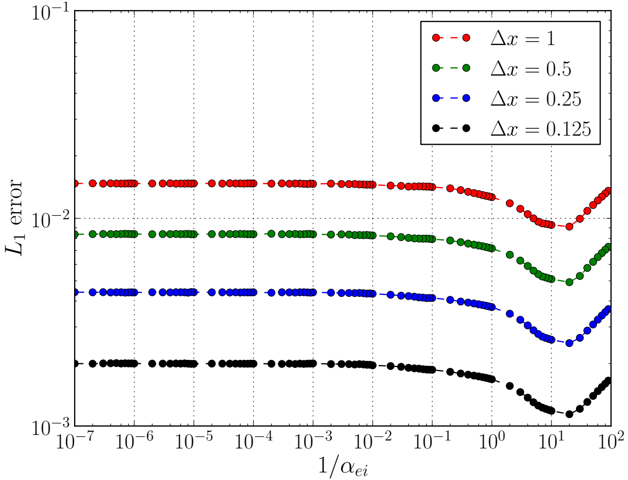


Figure: L_1 error as function of $1/\alpha_{ei}$ in log scale (Relaxation of a temperature profile at time $t = 10$).

Limit of the M_1 model

$\hookrightarrow$ Perturbative analysis (linearisation) and comparison with Vlasov.

- Linearisation around an equilibrium state

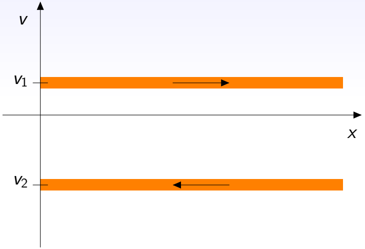
$$f_0(t,x,\zeta) = F_0(\zeta) + \delta F0(t,x,\zeta),$$
$$f_1(t,x,\zeta) = F_1(\zeta) + \delta F1(t,x,\zeta).$$

- Space and time Fourier transform

$$\hat{f}(\omega,k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,x)e^{i(\omega t - kx)}dxdt.$$

$\hookrightarrow$ dispersion relation derivation

Electron beams interaction (different velocities)



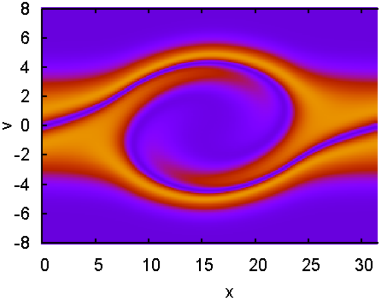
Relevance for laser-plasma interaction

Initial conditions:

$$f(t=0,x,v) = \frac{1}{2}(1 + A\cos(kx))\exp(-(v+v_1)^2) + \frac{1}{2}(1 - A\cos(kx))\exp(-(v+v_2)^2),$$

$$E(0,x) = 0.$$

$\hookrightarrow$ Correct dispersion relation

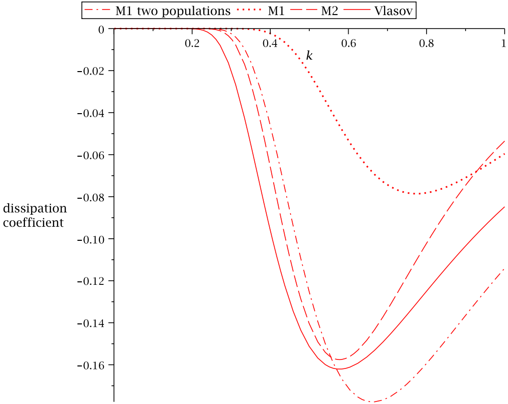


Landau damping

Interest in plasma physics and galaxy dynamics

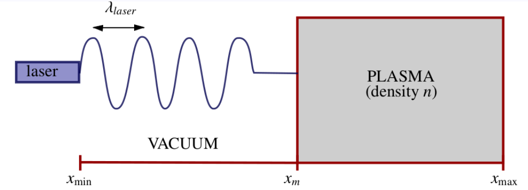
$\hookrightarrow$ Perturbation of an isotropic Maxwellian

- Two population M_1 model: $f = f^+ + f^-$
- M_2 model: higher order moments model



Laser-plasma absorption (collisionless skin effect)

Electromagnetic configuration



↪ Absorption coefficient derivation [37]

↪ M_1 model: not able to see the absorption phenomenon.

↪ Limit of the two populations M_1 and M_2 models[38].

[37] W. Rozmus, V. T. Tikhonchuk and R. Cauble. Phys. of Plasmas (1996).

[38] S. Guisset, J.G. Moreau, R. Nuter, S. Brull, E. d'Humières, B. Dubroca, V.T. Tikhonchuk. J. Phys. A: Math. Theor. (2015).